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LETTER TO THE EDITOR

Emptiness formation probability of the XXZ spin-$\frac{1}{2}$ Heisenberg chain at $\Delta = \frac{1}{2}$

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Abstract

Using a multiple integral representation for the correlation functions, we compute the emptiness formation probability of the XXZ spin-$\frac{1}{2}$ Heisenberg chain at anisotropy $\Delta = \frac{1}{2}$. We prove that it is expressed in terms of the number of alternating sign matrices.

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The Hamiltonian of the XXZ spin-$\frac{1}{2}$ Heisenberg chain is given by

$$H = \sum_{m=1}^{M} \left(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta \left(\sigma_m^z \sigma_{m+1}^z - 1\right)\right). \tag{1}$$

Here Δ is the anisotropy parameter, $\sigma_m^{x,y,z}$ denote the usual Pauli matrices acting on the quantum space at site m of the chain. The emptiness formation probability $\tau(m)$ (the probability of finding in the ground state a ferromagnetic string of length m) is defined as the following expectation value:

$$\tau(m) = \langle \psi_g | \prod_{k=1}^{m} \frac{1 - \sigma_k^z}{2} | \psi_g \rangle \tag{2}$$

where $|\psi_g\rangle$ denotes the normalized ground state. In the thermodynamic limit ($M \to \infty$), this quantity can be expressed as a multiple integral with m integrations [1–5]. Recently, in [6], a new multiple integral representation for $\tau(m)$ was obtained; for $\Delta = \cos\zeta$, $0 < \zeta < \pi$, one has

$$\tau(m) = \lim_{\xi_1,\ldots,\xi_m \to -\frac{i\zeta}{2}} \tau(m, \{\xi_j\}) \tag{3}$$

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where

$$\tau(m,\{\xi_j\}) = \frac{1}{m!}\int_{-\infty}^{\infty} \frac{Z_m(\{\lambda\},\{\xi\})}{\prod_{a<b}^m \sinh(\xi_a-\xi_b)} \det_m\left(\frac{\mathrm{i}}{2\zeta\sinh\frac{\pi}{\zeta}(\lambda_j-\xi_k)}\right) \mathrm{d}^m\lambda \tag{4}$$

with

$$Z_m(\{\lambda\},\{\xi\}) = \prod_{a=1}^m\prod_{b=1}^m \frac{\sinh(\lambda_a-\xi_b)\sinh(\lambda_a-\xi_b-\mathrm{i}\zeta)}{\sinh(\lambda_a-\lambda_b-\mathrm{i}\zeta)} \frac{\det_m\left(\frac{-\mathrm{i}\sin\zeta}{\sinh(\lambda_j-\xi_k)\sinh(\lambda_j-\xi_k-\mathrm{i}\zeta)}\right)}{\prod_{a>b}^m \sinh(\xi_a-\xi_b)}. \tag{5}$$

In this letter, we consider the particular case $\Delta = \frac{1}{2}$ ($\zeta = \pi/3$). Recently several interesting conjectures were obtained for the ground state of the model at this special value of the anisotropy parameter Δ [7–10]. Note that the unitary transformation $UH_\Delta U^{-1} = -H_{-\Delta}$, $U = \prod_{j=1}^{M/2}\sigma_{2j}^z$ relates our Hamiltonian (1) for $\Delta = \frac{1}{2}$ to the case $\Delta = -\frac{1}{2}$ in [8]. In particular, it was conjectured in [8] that, in this case, the emptiness formation probability is equal to

$$\tau(m) = \left(\frac{\sqrt{3}}{2}\right)^{3m^2} \prod_{k=1}^m \frac{\Gamma\left(k-\frac{1}{3}\right)\Gamma\left(k+\frac{1}{3}\right)}{\Gamma\left(k-\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}. \tag{6}$$

The aim of this letter is to give the proof of this conjecture using representations (3)–(5). We observe first that for $\zeta = \pi/3$,

$$Z_m(\{\lambda\},\{\xi\}) = \frac{(-1)^{\frac{m^2-m}{2}}}{2^{m^2+m}} \prod_{a>b}^m \frac{\sinh 3(\xi_b-\xi_a)}{\sinh(\xi_b-\xi_a)\sinh(\xi_a-\xi_b)}$$
$$\times \det_m\left(\frac{1}{\sinh(\lambda_j-\xi_k)\sinh(\lambda_j-\xi_k-\mathrm{i}\zeta)}\right) \frac{\det_m\left(\frac{1}{\sinh(\lambda_j-\xi_k+\frac{\mathrm{i}\pi}{3})}\right)}{\det_m\left(\frac{1}{\sinh 3(\lambda_j-\xi_k)}\right)}. \tag{7}$$

Here we have used the identities

$$\det_n \frac{1}{\sinh(x_j-y_k)} = \frac{\prod_{j>k}^n \sinh(x_j-x_k)\sinh(y_k-y_j)}{\prod_{j,k=1}^n \sinh(x_j-y_k)} \tag{8}$$

and $\sinh(3x) = 4\sinh(x)\sinh(x+\mathrm{i}\pi/3)\sinh(x-\mathrm{i}\pi/3)$. Substituting (7) into (4), we obtain

$$\tau(m,\{\xi_j\}) = \left(\frac{3\mathrm{i}}{4\pi}\right)^m \frac{(-1)^{\frac{m^2-m}{2}}}{2^{m^2}m!} \prod_{a>b}^m \frac{\sinh 3(\xi_b-\xi_a)}{\sinh(\xi_b-\xi_a)} \prod_{\substack{a,b=1\\a\neq b}}^m \sinh^{-1}(\xi_a-\xi_b)$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}^m\lambda \, \det_m\left(\frac{1}{\sinh\left(\lambda_j-\xi_k+\frac{\mathrm{i}\pi}{3}\right)}\right)$$
$$\times \det_m\left(\frac{1}{\sinh(\lambda_j-\xi_k)\sinh\left(\lambda_j-\xi_k-\frac{\mathrm{i}\pi}{3}\right)}\right). \tag{9}$$

Due to the symmetry properties of the integrand, we can replace the first determinant with the product of its diagonal elements multiplied by $m!$. Then, we insert each of these diagonal elements into the corresponding line of the second determinant. By this procedure, the integrals over the variables λ are decoupled and we can integrate each line of the determinant separately. Let us set $\xi_k = \varepsilon_k - \mathrm{i}\pi/6$. We obtain

$$\tau(m, \{\varepsilon_j\}) = (-1)^{\frac{m^2-m}{2}} 3^m 2^{-m^2} \prod_{a>b}^{m} \frac{\sinh 3(\varepsilon_b - \varepsilon_a)}{\sinh(\varepsilon_b - \varepsilon_a)} \prod_{\substack{a,b=1 \\ a\neq b}}^{m} \frac{1}{\sinh(\varepsilon_a - \varepsilon_b)}$$

$$\times \det_m \left(\int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{4\pi \cosh(\lambda - \varepsilon_j) \sinh\left(\lambda - \varepsilon_k - \frac{\mathrm{i}\pi}{6}\right) \sinh\left(\lambda - \varepsilon_k + \frac{\mathrm{i}\pi}{6}\right)} \right). \quad (10)$$

The computation of the integral over λ in (10) leads to

$$\tau(m, \{\varepsilon_j\}) = \frac{(-1)^{\frac{m^2-m}{2}}}{2^{m^2}} \prod_{a>b}^{m} \frac{\sinh 3(\varepsilon_b - \varepsilon_a)}{\sinh(\varepsilon_b - \varepsilon_a)} \prod_{\substack{a,b=1 \\ a\neq b}}^{m} \frac{1}{\sinh(\varepsilon_a - \varepsilon_b)} \det_m \left(\frac{3 \sinh \frac{\varepsilon_j - \varepsilon_k}{2}}{\sinh \frac{3(\varepsilon_j - \varepsilon_k)}{2}} \right). \quad (11)$$

To obtain the emptiness formation probability (2), one has to take the homogeneous limit $\varepsilon_j \to 0$. Using the fact that

$$\lim_{\substack{x_j \to x \\ y_k \to y}} \frac{\det_m f(x_j - y_k)}{\prod_{a>b}^{m} (x_a - x_b)(y_b - y_a)} = \prod_{n=0}^{m-1} (n!)^{-2} \det_m \left[f^{(j+k-2)}(z) \right] \qquad z = x - y \quad (12)$$

we finally obtain

$$\tau(m) = (-1)^{\frac{m^2-m}{2}} 3^{\frac{m^2+m}{2}} 2^{-m^2} \prod_{n=0}^{m-1} (n!)^{-2} \det_m \left[\frac{\partial^{j+k-2}}{\partial x^{j+k-2}} \frac{\sinh \frac{x}{2}}{\sinh \frac{3x}{2}} \right]_{x=0}. \quad (13)$$

The determinant in (13) can be computed using the following identity [11]:

$$\frac{1}{\prod_{j>k}^{m} \sinh^2 \beta(j-k)} \det_m \frac{\sinh \alpha(j+k-1)}{\sinh \beta(j+k-1)} = 2^{m^2-m} \prod_{j=1}^{m} \prod_{k=1}^{m} \frac{\sinh(\alpha + \beta(j-k))}{\sinh \beta(j+k-1)} \quad (14)$$

(see [11] for the proof). The determinant (13) is a particular case of (14). Indeed, we can consider the case $\beta = 3\alpha$, $\alpha \to 0$, and apply (12) for $x_j = \alpha j$, $y_k = \alpha(1-k)$. Then,

$$\prod_{n=0}^{m-1} (n!)^{-2} \det_m \left[\frac{\partial^{j+k-2}}{\partial x^{j+k-2}} \frac{\sinh \frac{x}{2}}{\sinh \frac{3x}{2}} \right]_{x=0} = 3^{m^2-m} \prod_{j=1}^{m} \prod_{k=1}^{m} \frac{j - k + \frac{1}{3}}{j + k - 1}$$

$$= (-1)^{\frac{m^2-m}{2}} 3^{-\frac{m+m^2}{2}} \prod_{k=0}^{m-1} \frac{(3k+1)!}{(m+k)!}. \quad (15)$$

Substituting these expressions into (13), we finally obtain

$$\tau(m) = \left(\frac{1}{2}\right)^{m^2} \prod_{k=0}^{m-1} \frac{(3k+1)!}{(m+k)!}. \quad (16)$$

We observe that the quantity $A_m = \prod_{k=0}^{m-1} (3k+1)!/(m+k)!$ is the number of alternating sign matrices of size m [12]. Using

$$\Gamma(3z) = \frac{1}{2\pi} 3^{3z-1/2} \Gamma(z)\Gamma(z+1/3)\Gamma(z+2/3) \qquad \Gamma(k+1/2) = \frac{\sqrt{\pi}}{2^k}(2k-1)!! \quad (17)$$

one can easily check the equivalence of (16) and (6). Thus (6) is proved.

The asymptotic behaviour of $\tau(m)$ for $m \to \infty$ can be evaluated using the Stirling formula [8]:

$$\tau(m) \to c \left(\frac{\sqrt{3}}{2}\right)^{3m^2} m^{-\frac{5}{36}} \qquad m \to \infty \quad (18)$$

with

$$c = \exp\left[\int_0^\infty \left(\frac{5\mathrm{e}^{-t}}{36} - \frac{\sinh\frac{5t}{12}\sinh\frac{t}{12}}{\sinh^2\frac{t}{2}}\right)\frac{\mathrm{d}t}{t}\right]. \qquad (19)$$

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